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Quantization of Galilean Systems

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Abstract

The Kostant quantization method is applied to the Galilean group.

1. *Introduction*

The abstract mathematical setting for studying classical mechanics is that of a symplectic manifold (X, Ω). X is the phase space and Ω defines the Poisson bracket (Abraham, 1967). The requirement of invariance under a Lie group G of space-time transformations leads directly to the study of Lie group actions on symplectic manifolds (X, Ω) that leave invariant the form Ω. Following Arens (1971), a classical system (X, Ω) is considered to be elementary if the action of G on X is transitive. Kostant (1970) and Renouard (1969) have shown how to classify such elementary systems, and Kostant (1969) has shown that often they can be quantized to obtain a unitary irreducible representation of the group G. This paper is a study of Kostant's method when G is the Galilean group. The Poincaré group has been considered in this context by Renouard (1969). For Galilean systems the central extensions of G arise as naturally, and play the same role in classical mechanics as in quantum mechanics. That is, they introduce the mass and give rise to the only systems with a clear particle interpretation. The quantization of these systems is shown to be the projective unitary irreducible representations of G, which are well known (Levy-Leblond, (1963) to describe the free Schroedinger particles. Souriau (1970) has obtained this result by his own slightly different quantization technique. The symplectic manifolds associated with the Galilean group itself are shown to give rise to all the true representations classified I-IV by Inönü and Wigner (1952). Some of these have recently been shown (Sen, 1973) to have an unusual physical application.

2. *Notation and General Theory*

If G is any Lie group with multiplier group $H^2(G, T)$, then G_σ denotes the central extension of G, by the circle group T, corresponding to the multiplier σ in $H^2(G, T)$. In the language of Kostant (1969) an elementary classical G system is defined to be a strongly symplectic homogenous G-space. Renouard (1969) has shown that such spaces cover an orbit of the coadjoint action of one of the G_σ in the dual $\mathfrak{g}_\sigma^*$ of its Lie algebra $\mathfrak{g}_\sigma$. The G actions are obtained directly from the G_σ actions. Having obtained the classical systems (X, Ω) as the orbits of the G_σ actions in $\mathfrak{g}_\sigma^*$, their quantization may be achieved by either of two methods.

One approach is to construct a line bundle L over X with curvature Ω and then to define a Hilbert space in which G_σ acts unitarily by taking certain sections of L. The other (Kostant, 1969) is to construct a sequence of subgroups of G_σ and apply Mackey's induction theory (Mackey, 1968) to obtain a unitary representation of G_σ. This second method is described briefly below and applied later in the work.

The coadjoint action of a Lie group G on $\mathfrak{g}^*$, written $g \circ a^*$. for $g \in G$ and $a^* \in \mathfrak{g}^*$, is defined by

$$\langle g \circ a^*, a \rangle = \langle a^*, \mathrm{Ad}(g^{-1}) a \rangle$$

where $a \in \mathfrak{g}$, Ad is the adjoint representation of G and $\langle\ ,\ \rangle$ is a pairing of $\mathfrak{g}$ with $\mathfrak{g}^*$. Each orbit of this action is diffeomorphic as a G-space to G/K, where K is its stability group. Let a be a fixed point in $\mathfrak{g}^*$ with stability group K; then an invariant polarization for the orbit through a^* is a complex subalgebra $\mathfrak{h}$ of the complexification $\mathfrak{g}^{\mathbb{C}}$ of $\mathfrak{g}$ with the properties (i) $\mathfrak{k} \subset \mathfrak{h}$ where $\mathfrak{k}$ is the Lie algebra of K, (ii) $\mathfrak{k}$ is $\mathrm{Ad}(K)$ stable, (iii) $\dim_{\mathbb{C}} \mathfrak{g}^{\mathbb{C}}/\mathfrak{g} = \frac{1}{2} \dim_{\mathbb{R}} \mathfrak{g}/\mathfrak{k}$, where $\dim_{\mathbb{F}}$ means dimension with respect to the field $\mathbb{F}$, (iv) $\langle a^*, [\mathfrak{h}, \mathfrak{h}]\rangle = 0$, (v) $\mathfrak{h} + \bar{\mathfrak{h}}$ is a subalgebra of $\mathfrak{g}^{\mathbb{C}}$. A polarization $\mathfrak{h}$ defines two subalgebras $\mathfrak{d} = \mathfrak{h} \cap \mathfrak{g}$ and $\mathfrak{e} = (\mathfrak{h} + \bar{\mathfrak{h}}) \cap \mathfrak{g}$, which in turn define connected subgroups D_0 and E_0 of G. D_0 and E_0 are normalized by K, so that $D = D_0 K$ and $E = E_0 K$ are also subgroups of G. Clearly $K \subset D \subset E \subset G$. This is the sequence of groups alluded to above. Now $2\pi \sqrt{-1} a^*$ is a homomorphism from $\mathfrak{k}$ to the Lie algebra $\sqrt{-1}\, \mathbb{R}$ of T. The integrality condition (Kostant, 1969) requires that this homomorphism should lift to a character χ of K. χ is then extended, in a unique way, to a character of D which is induced holomorphically to E. The final step is to induce from E to G in the usual way. It is not always the case that χ is extendable to D or that the representation of G, obtained in this way, is irreducible.

3. *The Galilean Group*

From this point on G is the Galilean group. Its elements g are the quadruples (W, b, v, u), where $W \in SO(3, \mathbb{R})$, $b \in \mathbb{R}$, $v \in \mathbb{R}^3$, $u \in \mathbb{R}^3$. Now the product gg' is $(WW', b + b', Wv' + v, Wu' + u + b'v)$, where $g' = (W', b', v', u')$, giving G the semidirect product structure $[SO(3,\mathbb{R}) \times \mathbb{R}] \otimes \mathbb{R}^6$. Bargmann (1954) has shown that $H^2(G, T) \cong \mathbb{R}$. The following form may be chosen for the multipliers: $\sigma_m(g, g') = \exp\sqrt{-1}m(\frac{1}{2}b'v^2 + Wu' \cdot v)$, where m is a real parameter that changes the equivalence class in $H^2(G, T)$. The multiplication laws for the G_σ are $(g, t)(g', t') = (gg', \sigma(g, g')tt')$ where t and t' are in T'.

As a vector space $\mathfrak{g}$ is isomorphic to $\mathfrak{so}(3, \mathbb{R}) \otimes \mathbb{R}^7$ and each element a of $\mathfrak{g}$ can be written as a quadruple (w, b, v, u) for $w \in \mathfrak{so}(3, \mathbb{R})$, $b \in \mathbb{R}$, $v \in \mathbb{R}^3$, $u \in \mathbb{R}^3$, in terms of which the Lie bracket is $[a, a'] = (ww' - w'w, 0, wv' - w'v, wu' - w'u + b'v - bv')$ for $a' = (w', b', v', u')$. Defining e_i ($i = 1, 2, 3$) to be the unit vector in the ith direction of $\mathbb{R}^3$, the following basis for $\mathfrak{g}$ is adopted:

$$j_{0i} = (J_i, 0, 0, 0), \quad b_0 = (0, 1, 0, 0)$$
$$v_{0i} = (0, 0, e_i, 0), \quad u_{0i} = (0, 0, 0, e_i)$$

J_i ($i = 1, 2, 3$) being the basis of $\mathfrak{so}(3, \mathbb{R})$ with the property $\Lambda(J_i, J_k) = 2\delta_{ik}$ with respect to the Killing form Λ on $\mathfrak{so}(3, \mathbb{R})$. If elements a^* of $\mathfrak{g}^*$ are written (w^*, b^*, v^*, u^*) and $\mathfrak{g}^*$ is identified with $\mathfrak{so}(3, \mathbb{R}) \otimes \mathbb{R}^7$ by the pairing

$$\langle (w^*, b^*, v^*, u^*), (w, b, v, u) \rangle = \tfrac{1}{2}\Lambda(w^*, w) + v^* \cdot v + b^* b + u^* \cdot u$$

then the coadjoint action of G on $\mathfrak{g}^*$ is $g \circ a^* = (Ww^*W^{-1} + Wv^* \wedge v + Wu^* \wedge u, b^* - Wu^* \cdot v, Wv^* + bWu^*, Wu^*)$ where for $x \in \mathbb{R}^3$ and $y \in \mathbb{R}^3$, $x \wedge y$ is the element of $\mathfrak{so}(3, \mathbb{R})$ given by $(x \wedge y)z = x \cdot zy - y \cdot zx$ for $z \in \mathbb{R}^3$. The adjoint representation of G being

$$\mathrm{Ad}(g)a = (WwW^{-1}\,b, Wv - WwW^{-1}v, WwW^{-1}(bv - u) - bWv + Wu + vb)$$

Four classes of orbit in $\mathfrak{g}^*$ are now considered; $j^*_{0i}, b^*_0, v^*_{0i}, u^*_{0i}$ ($i = 1, 2, 3$) is the basis of $\mathfrak{g}^*$ dual to that of $\mathfrak{g}$ with respect to the pairing $\langle\ ,\ \rangle$.

(i) As representatives of orbits in this class the points $a^* = \lambda v^*_{02} + \mu u^*_{03}$ may be taken. Each pair of positive numbers λ and μ defines an orbit. The stability group K is the set of elements $(I, 0, (0, v_2, 0), (0, 0, u_3))$ of G, where v_2 and u_3 are arbitrary and I is the identity of $SO(3, \mathbb{R})$. Hence the topology of these orbits is that of $SO(3, \mathbb{R}) \times \mathbb{R}^5$. The subalgebra $\mathfrak{h}$ with the basis $v_{01}, v_{02}, v_{03}, u_{01}, u_{02}, u_{03}$ is easily shown to satisfy the conditions of an invariant polarization for this class of orbits. $\mathfrak{h}$ defines the subgroups D and E to be $D = E = \{(I, 0, v, u)\colon v \in \mathbb{R}^3, u \in \mathbb{R}^3\}$. The integrality condition is satisfied without restriction on λ and μ

(ii) Representatives for these orbits may be taken as $a^* = \rho j^*_{03} + \mu u^*_{03}$, obtaining each orbit once for real ρ and positive μ. The stability group K is $SO(2, \mathbb{R}) \otimes \{(v_1, v_2, 0), (0, 0, u_3)\}$ where v_1, v_2, and u_3 are arbitrary. It follows that each orbit is topologically $S^2 \times \mathbb{R}^4$. An invariant polarization $\mathfrak{h}$ for this class is given by the basis elements $j_{03}, v_{01}, v_{02}, v_{03}, u_{01}, u_{02}, u_{03}$. Thus $D = E = SO(2, \mathbb{R}) \otimes R^6$. Integrality requires that $2\pi\rho$ be an integer.

(iii) For real ρ and ϵ and positive μ this class comprises the orbits through the points $a^* = \rho j^*_{03} + \epsilon b^*_0 + \mu u^*_{03}$. K is the group $(SO(2, \mathbb{R}) \times \mathbb{R}) \otimes \{(0, 0, v_3), u\}$ with $v_3 \in \mathbb{R}$ and $u \in \mathbb{R}^3$. These orbits have the topology of $S^2 \times \mathbb{R}^2$. $\mathfrak{h}$ can be defined by the basis elements $j_{03}, b_0, v_{01}, v_{02}, v_{03}, u_{01}, u_{02}, u_{03}$, which implies that $D = E = (SO(2, \mathbb{R}) \times \mathbb{R}) \otimes \mathbb{R}^6$. $2\pi\rho$ must be an integer for the integrality condition to be satisfied.

(iv) $a^* = \rho j^*_{03} + \epsilon b^*_0$ for real ρ and ϵ are taken as representatives for this class of orbits. K is the group $(SO(2, \mathbb{R}) \times \mathbb{R}) \otimes \mathbb{R}^6$ and each orbit has the topology of S^2. The basis

$$j_{03}, j_{01} + \sqrt{-1} j_{02}, b_0, v_{01}, v_{02}, v_{03}, u_{01}, u_{02}, u_{03}$$

defines an invariant polarization. $D = E$ and $E = G$. Again the integrality condition for a^* requires $2\pi\rho$ to be an integer.

For each nontrivial σ in $H^2(G, T)$ $\mathfrak{g}_\sigma$ is the vector space isomorphic to $\mathfrak{g} \otimes \sqrt{-1}\mathbb{R}$. A pairing $\langle \ , \ \rangle_\sigma$ of $\mathfrak{g}^*_\sigma$ with $\mathfrak{g} \otimes \sqrt{-1}\mathbb{R}$ can be defined by

$$\langle \{a^*, \sqrt{-1} f^*\}, \{a, \sqrt{-1} f\} \rangle_\sigma = \langle a^*, a \rangle - f^* f$$

in terms of which the coadjoint action of G_σ on $\mathfrak{g}^*_\sigma$, when $\sigma = \sigma_m$ is

$$(g, t) \circ \{a^*, \sqrt{-1} f^*\} = \{g \circ a^* + m f^* (-v \wedge u, \tfrac{1}{2} v^2, -v, u - bv), \sqrt{-1} f^*\}$$

The adjoint representation of G_σ being

$$\mathrm{Ad}(g, t)\{a, \sqrt{-1} f\} = \{\mathrm{Ad}(g)a, \sqrt{-1}(f + \tfrac{1}{2} m b v^2$$
$$+ mb(W\omega W^{-1} v) \cdot v - m(W\omega W^{-1} u) \cdot v - mu \cdot Wv)$$

The basis of $\mathfrak{g}$ given above is augmented by the element $\ell_0 = (0, 0, 0, 0, \sqrt{-1})$ to form a basis for $\mathfrak{g}_\sigma$. Its dual element is denoted ℓ^*_0. Only two topologically distinct classes of orbit occur:

(a) The first class consists of those through the points $\omega j^*_{03} + \epsilon b^*_0 + (\tau/2\pi)\ell^*_0$, where ω and τ are nonzero and ϵ is arbitrary. The stability group for this class is $K = SO(2, \mathbb{R}) \times \mathbb{R} \times T$ and the topology of the orbits is that of $S^2 \times \mathbb{R}^6$. The basis $j_{01}, j_{02} + \sqrt{-1} j_{03}, b_0, v_{01}, v_{02}, v_{03}$ defines a suitabile invariant polarization. $D = (SO(2, \mathbb{R}) \times \mathbb{R}) \otimes \{(v, 0, t)\}$ and $E = (SO(3, \mathbb{R}) \times \mathbb{R}) \otimes \{(v, 0, t)\}$, where $v \in \mathbb{R}^3$ and $t \in T$. Integrality requires $2\pi\omega$ and τ to be integer[5].

(b) The second class consists of those through the points $\epsilon b^*_0 + (\tau/2\pi)\ell^*_0$, for nonzero τ and arbitrary ϵ. $K = SO(3, \mathbb{R}) \times \mathbb{R} \times T$, hence the orbits are topologically $\mathbb{R}^6$. An invariant polarization can be defined by the basis $j_{01}, j_{02}, j_{03}, b_0, v_{01}, v_{02}, v_{03}$, which gives $D = E = (SO(3, \mathbb{R}) \times \mathbb{R}) \otimes \{(v, 0, t)\}$. τ has to be an integer for the integrality condition to be satisfied.

4. *Discussion and Conclusion*

Comparison with the paper of Inonu and Wigner (1952) reveals that the classes (i)–(iv) of representations above correspond to their classes I–IV; in particular all unitary irreducible representations of G arise by quantizing orbits. The representations in classes (a) and (b) correspond, when $\tau = -1$, to the projective σ_m representations of G, those with spin zero being in (a). The parameter ϵ in (a) and (b) is the arbitrary constant of energy well known in the quantum case not to alter the equivalence class of the projective representation (Levy-Leblond, 1963). That equivalent classical systems are obtained by choosing ϵ arbitrarily is immediate from a theorem of Renouard (1969).

The only symplectic forms that give rise to the usual Poisson bracket belong to those orbits in class (b) for which $\tau = -1$. Coordinates $(p, q); p \in R^3, q \in R^3$ can be defined for this case that transform, respectively, as momentum and position under the G actions. In these coordinates the form Ω is $\Sigma_i dp_i \times dq_i$. The generator of time translations can be computed as

$$\frac{1}{2m} \sum_i p_i^2 + \epsilon$$

which identifies these systems as the free Newtonian particles with mass m. The symplectic forms for the orbits in (b) are as in (a) plus a form on S^2; these systems may be interpreted as classical nonrelativistic particles with spin (Arens, 1971). The forms for the orbit classes (i)–(iii) do not separate (in natural coordinates) into a form on the vector part plus a form on the rest. For example in (iii) they are given locally by

$$\Omega = (\rho/s_3) ds_1 \times ds_2 + \mu(ds_1 \times dx + ds_2 \times dy)$$

where (s_1, s_2, s_3) are cartesian coordinates for S^2 and (x, y) are coordinates for $\mathbb{R}^2$. These systems lack interpretation as free particles, as do their quantizations (Inönü and Wigner 1952).

The polarizations given are not the only ones possible. For example the basis $j_{03}, \ell_0, v_{01}, v_{02}, v_{03}, u_{02}, u_{01}$ defines a further invariant polarization for the class (ii) orbits. In fact this polarization gives a sequence of groups that corresponds to applying Mackey's semidirect product theory to G expressed as $E(3) \otimes \mathbb{R}^4$, where $E(3)$ is the Euclidean group in three dimensions, instead of of $(SO(3, \mathbb{R}) \times \mathbb{R}) \otimes \mathbb{R}^6$.

The half-integer spin projective representations with nonzero mass may be obtained by replacing $SO(3, \mathbb{R})$ by its cover $SU(2, \mathbb{C})$, as may the projective zero-mass representations.

Appendix

The following are the properties of the Killing form Λ and the $\wedge$ product used to derive the coadjoint actions quoted in the text:

(1) $[\omega, x \wedge y] = \omega x \wedge y + x \wedge \omega$ for $\omega \epsilon SO(3, R)$, and $x \in R^3, y \in R^3$.
(2) $\Lambda(\omega_1, \omega_2) = \mathrm{Tr}(\omega_1 \omega_2)$ for ω_1 and $\omega_2 \in SO(3, R)$
(3) $\mathrm{Tr}(\omega x \wedge y) = 2x \cdot \omega y$
(4) $\Lambda(\omega, x \wedge y) = 2x \cdot \omega y$

The following theorem is that of Renouard, referred to in text:

> *Théorème.* Si (X, Ω) est un G-espace homogene symplectique, G est fortement symplectique sur (X, Ω) si et seulement si il existe une

G-orbite $Y \subset \mathfrak{g}_0^*$ telle que (X, Ω) soit un revetement de (Y, ω_y) comme G-espaces homogenes symplectiques.

Si $Y' \subset \mathfrak{g}_0^*$ est une autre orbite possedant la meme propriete, on a $Y' = f + Y\tau'(x) = f + \tau(x)$ pour tout $x \in X$, avec $f \in \hat{\mathfrak{g}}$ où $\hat{\mathfrak{g}} = \{f \in \mathfrak{g}^*, \langle f, [\mathfrak{g}, \mathfrak{g}] \rangle = 0\}$

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